CALCULATION OF RADIATIVE HEAT TRANSMISSION THROUGH DISPERSIVE MEDIA

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A method is outlined for solving the equation of energy radiation and also for determining the thermal radiation flux in an emitting, an absorbing, and an anisotropically dispersing medium. Values of radiation flux calculated here agree closely with data published in the technical literature.

Calculating the radiation in a gaseous medium with a slight admixture of solid particles, liquid droplets, or opaque gases is worthwhile for the evaluation of many high-temperature processes. In order to calculate the thermal radiation flux in such contaminated media, it is necessary to solve the integrodifferential equation of heat radiation. Under conditions of local thermodynamic equilibrium and with the intensity of monochromatic radiation independent of the azimuth angle φ, this equation for a plane layer of an emitting, an absorbing, and a dispersing medium is

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = (1-\omega_0) I_b(\tau) + \frac{\omega_0}{2} \int_{-1}^{1} P(\mu, \mu') I(\tau, \mu') d\mu', \quad (1)$$

where function $P(\mu, \mu')$ has been obtained from the dispersion indicatrix $P(\theta) = \sum_{i=0}^{l} a_i P_i(\cos\theta)$ at the angle subtending the layer, according to the law of cosines in spherical trigonometry

$$\cos\theta = \mu\mu' + (1-\mu^2)^{1/2} [1-(\mu')^2]^{1/2} \cos(\varphi - \varphi')$$

and by subsequent integration with respect to $(\varphi - \varphi')$ [2]:

$$P(\mu, \mu') = \sum_{i=0}^{l} a_i P_i(\mu) P_i(\mu'). \quad (2)$$

We will assume that the plane layer of dispersing material is bounded by plane-parallel diffusively radiating surfaces. The boundary conditions at these surfaces will then be [3]:

$$I(0, \mu)|_{\mu>0} = \varepsilon_1 I_b(0) + 2\rho_1 \int_0^{-1} I(0, \mu)|_{\mu<0}\, \mu d\mu, \quad (3)$$

$$I(\tau_0 \mu)|_{\mu<0} = \varepsilon_2 I_b(\tau_0) + 2\rho_2 \int_0^{1} I(\tau_0, \mu)|_{\mu>0}\, \mu d\mu.$$

Since the radiation intensity in problems of this kind is discontinuous at $\mu = 0$, hence it is worthwhile to split the problem into $I^+(\tau, \mu)$ with $0 < \mu \leq 1$ and $I^-(\tau, \mu)$ with $-1 \leq \mu < 0$, and then, following the Eavon method [1], to use the Legendre expansion of $I^+(\tau, \mu)$ into polynomials $P_n(2\mu-1)$ and of $I^-(\tau, \mu)$ into polynomials $P_n(2\mu+1)$:

$$I^+(\tau, \mu) = \frac{1}{4\pi} \sum_{i=0}^{\infty} (2i+1) I_i^+(\tau) P_i(2\mu - 1),$$

Institute of Heat and Mass Transfer, Academy of Sciences of the USSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 5, pp. 803-812, May, 1973. Original article submitted September 14, 1972.

$$I^-(\tau, \mu) = \frac{1}{4\pi}\sum_{i=0}^{\infty}(2i+1) I_i^-(\tau) P_i(2\mu+1)$$
$$(i = 0, 1, \ldots), \tag{4}$$

where

$$I_i^+(\tau) = 2\pi\int_{-1}^{1} I^+\left(\tau, \frac{\mu+1}{2}\right) P_i(\mu)\, d\mu, \; I_i^-(\tau) = 2\pi\int_{-1}^{1} I^-\left(\tau, \frac{\mu-1}{2}\right) P_i(\mu)\, d\mu.$$

In these expansions (4) we will retain only N terms, and this number N will determine the degree of approximation to which the radiation equation is solved by that method.

Inserting (4) into (1), we obtain two equations for $I^+(\tau, \mu)$ and $I^-(\tau, \mu)$ (here and henceforth we imply summation over i)

$$(2i+1)\mu P_i(2\mu-1)\frac{dI_i^+(\tau)}{d\tau} + (2i+1) P_i(2\mu-1) I_i^+(\tau) = 4\pi(1-\omega_0) I_b(\tau)$$
$$+\frac{\omega_0}{2}\Big[(2i+1) I_i^+(\tau)\int_0^1 P_i(2\mu-1) P(\mu, \mu')\, d\mu' + (2i+1) I_i^-(\tau) \times$$
$$\times \int_{-1}^{0} P_i(2\mu+1) P(\mu, \mu')\, d\mu',$$
$$(2i+1)\mu P_i(2\mu+1)\frac{dI_i^-(\tau)}{d\tau} + (2i+1) P_i(2\mu+1) I_i^-(\tau) = 4\pi(1-\omega_0) I_b(\tau)$$
$$+\frac{\omega_0}{2}\Big[(2i+1) I_i^+(\tau)\int_0^1 P_i(2\mu-1) P(\mu, \mu')\, d\mu' + (2i+1) I_i^-(\tau)$$
$$\times \int_{-1}^{0} P_i(2\mu'+1) P(\mu, \mu')\, d\mu'\Big]. \tag{5}$$

We next multiply equations (5) by $(2k+1)P_k(2\mu-1)$ and by $(2k+1)P_k(2\mu+1)$ respectively, then integrate with respect to μ the first equation from 0 to 1 and the second equation from −1 to 0. Letting $k = 0, 1, 2, \ldots$, we obtain a system of $2(N+1)$ ordinary differential equations in functions $I_i^+(\tau)$ and $I_i^-(\tau)$

$$\alpha_{ik}\frac{dI_i^+(\tau)}{d\tau} + \beta_{ik} I_i^+(\tau) = \frac{\omega_0}{2}\{I_i^+(\tau)\gamma_{+i+k} + I_i^-(\tau)\gamma_{-i+k}\} + 4\pi(1-\omega_0)\beta_{0k} I_b(\tau),$$
$$(-1)^{i+k+1}\alpha_{ik}\frac{dI_i^-(\tau)}{d\tau} + \beta_{ik} I_i^-(\tau) = \frac{\omega_0}{2}\{I_i^+(\tau)\gamma_{+i-k} + I_i^-(\tau)\gamma_{-i-k}\} +$$
$$+ 4\pi(1-\omega_0)\beta_{0k} I_b(\tau), \tag{6}$$

where

$$\alpha_{ik} = (2i+1)(2k+1)\int_0^1 \mu P_k(2\mu-1) P_i(2\mu-1)\, d\mu$$
$$= \begin{cases} \frac{i}{2}, & k = i-1, \\ \frac{2i+1}{2}, & k = i, \\ \frac{i+1}{2}, & k = i+1, \\ 0, & k < i-1, \; k > i+1 \end{cases} \tag{7}$$

and

$$\beta_{ik} = (2i+1)(2k+1)\int_0^1 P_k(2\mu-1) P_i(2\mu-1)\, d\mu$$
$$= (2i+1)(2k+1)\int_{-1}^{0} P_k(2\mu+1) P_i(2\mu+1)\, d\mu$$
$$= \begin{cases} 0, & i \neq k, \\ 2i+1, & i = k \end{cases} \tag{8}$$

by changing to variables $\bar{\mu} = 2\mu - 1$ or $\bar{\mu} = 2\mu + 1$ in the integrals and subsequently applying the condition of orthogonality of Legendre polynomials as well as the recurrence relation [4]

$$(n+1)P_{n+1}(\mu) + nP_{n-1}(\mu) = (2n+1)\mu P_n(\mu). \tag{9}$$

The coefficients matrix

$$\gamma_{+i+k} = (2i+1)(2k+1)\int_0^1 d\mu \int_0^1 P_k(2\mu-1)P_i(2\mu'-1)P(\mu, \mu')\,d\mu',$$

$$\gamma_{+i-k} = (2i+1)(2k+1)\int_{-1}^0 d\mu \int_0^1 P_k(2\mu+1)P_i(2\mu'-1)P(\mu, \mu')\,d\mu',$$

$$\gamma_{-i+k} = (2i+1)(2k+1)\int_0^1 d\mu \int_{-1}^0 P_k(2\mu-1)P_i(2\mu'+1)P(\mu, \mu')\,d\mu',$$

$$\gamma_{-i-k} = (2i+1)(2k+1)\int_{-1}^0 d\mu \int_{-1}^0 P_k(2\mu+1)P_i(2\mu'+1)P(\mu, \mu')\,d\mu' \tag{10}$$

has the following symmetry properties:

1. By virtue of definition (2), expressions (10) yield

$$\gamma_{+i+k} = \gamma_{+k+i}, \quad \gamma_{-i-k} = \gamma_{-k-i}. \tag{11}$$

2. From the normalization of the dispersion indicatrix

$$\int_{-1}^1 P(\mu, \mu')\,d\mu' = 2, \tag{12}$$

follows

$$\gamma_{+0+k} + \gamma_{-0+k} = (2k+1)\int_0^1 d\mu \int_{-1}^1 P_k(2\mu-1)P(\mu, \mu')\,d\mu'$$

$$= (2k+1)\int_{-1}^1 P_k(\mu)\,d\mu = \begin{cases} 2, & k=0, \\ 0, & k \geqslant 1, \end{cases} \tag{13}$$

and, analogously,

$$\gamma_{-0+k} + \gamma_{-0-k} = \begin{cases} 2, & k=0, \\ 0, & k \geqslant 1. \end{cases} \tag{14}$$

3. By virtue of definition (2) and the property of Legendre polynomials that $P_k(-\mu) = (-1)^k P_k(\mu)$, $P(\mu, \mu')$ does not change after a simultaneous replacement of $\mu \to -\mu$ and $\mu' \to -\mu'$, so that

$$\gamma_{-i-k} = (-1)^{i+k}\gamma_{+i+k}, \quad \gamma_{+i-k} = (-1)^{i+k}\gamma_{-i+k}. \tag{15}$$

The boundary conditions for $I_i^+(\tau)$ and $I_i^-(\tau)$ are derived by analogy to (6), i. e., by inserting the expansion (4) into the boundary conditions (3), then multiplying the first of them by $(2k+1)P_k(2\mu-1)$ and the second of them by $(2k+1)P_k(2\mu+1)$, then integrating from 0 to 1 and from -1 to 0 respectively, and thus obtaining $2(N+1)$ boundary conditions for $I_i^+(\tau)$ and $I_i^-(\tau)$:

$$\beta_{ik}I_i^+(0) = 4\pi\varepsilon_1 I_b(0) + 2\rho_1 I_i^-(0)\alpha_{i0}(-1)^i,$$
$$\beta_{ik}I_i^-(0) = 4\pi\varepsilon_2 I_b(\tau_0) + 2\rho_2 I_i^+(\tau_0)\alpha_{i0}. \tag{16}$$

Expansion (4) for $I^+(\tau, \mu)$ and $I^-(\tau, \mu)$ is convenient, inasmuch as it yields a rather simple expression for the thermal radiation flux q. Indeed,

$$q = 2\pi\int_{-1}^1 I(\tau, \mu)\mu\,d\mu = 2\pi\Big[\int_0^1 I^+(\tau, \mu)\mu\,d\mu - \int_0^{-1} I^-(\tau, \mu)\mu\,d\mu\Big]$$

$$= \frac{1}{2}\Big[\sum_{i=0}^{\infty} I_i^+(\tau)(2i+1)\int_0^1 P_i(2\mu-1)\mu\,d\mu - \sum_{i=0}^{\infty} I_i^-(\tau)(2i+1)$$

$$\times \int_0^{-1} P_i(2\mu+1)\mu\,d\mu\Big] = \frac{1}{4}\Big\{\big[I_0^+(\tau) - I_0^-(\tau)\big] + \big[I_1^+(\tau) - I_1^-(\tau)\big]\Big\}. \tag{17}$$

TABLE 1. Coefficients c_i in (32) for the Radiation Flux q

Simplest forward elongated dispersion indicatrix

τ_0	$c_1 \cdot 10^3$	c_2	$c_3 \cdot 10^2$	c_4
0.4	13.6	0,564	16,97	5,079
0.8	1.72	0,561	7,41	5,071
1.0	0.638	0,560	5,01	5,069
1.2	0.274	0,560	3,42	5,067
1.6	0.032	0,560	1,59	4,813
2.0	0.004	0,560	0.773	5,066

Spherical dispersion indicatrix

τ_0	c_1	c_2	c_3	c_4
0.1	0.085	0.619	0.532	4.783
0.5	0.009	0.607	0.219	4.727
1.0	0	0.603	0.146	4.719
2.0	0	0.591	0.071	4.701

All integrals under the summation sign Σ are equal to zero for $i \geq 2$.

It follows from the resulting expression (17) that, in order to calculate the exact thermal radiation flux, it suffices to know two coefficients in the Legendre polynomial expansion of radiation intensities $I^+ \cdot (\tau, \mu)$ and $I^-(\tau, \mu)$. For not too elongated dispersion indicatrices these coefficients can be found fairly accurately from the solution to system (6) with N = 1, i. e., to a system of fourth-order ordinary differential equations.

When the temperature distribution in the layer is known, then the solution of this system is easy. Otherwise, one must add to this system also the differential equation of energy and then the temperature distribution as well as the thermal flux distribution will have to be determined by differential methods of computation.

It is to be noted that, when N = 0, system (6) with the boundary condition (16) becomes identical to the well known differential-difference approximation used in [5] for axially symmetric indicatrices of any shape and for an isotropic intensity distribution in the flux. Indeed, by virtue of properties 1 to 3, we have

$$\gamma_{+0+0} = \gamma_{-0-0} = \gamma_1, \ \gamma_{+0-0} = \gamma_{-0+0} = \gamma_2$$

and

$$\gamma_{+0+0} + \gamma_{-0+0} = 2.$$

Coefficients γ_1 and γ_2 represent the fractions of radiation flux dispersed by a unit volume along its path of incidence, within the solid angle $2\pi_+$ and within the solid angle $2\pi_-$ respectively. As a consequence of the last relation, $\gamma_1 = 2$ corresponds to a dispersion indicatrix maximally elongated forward, $\gamma_1 = 1$ corresponds to a symmetric dispersion indicatrix, and $\gamma_1 = 0$ corresponds to a dispersion indicatrix maximally elongated backward.

In the conventional notation, system (6) for determining the thermal radiation flux $q = E_0^+(\tau) - E_0^-(\tau)$ becomes to the zeroth approximation

$$\frac{dE_0^+(\tau)}{d\tau} = (\omega_0\gamma_1 - 2)E_0^+(\tau) + \omega_0\gamma_2 E_0^-(\tau) + 2\pi(1-\omega_0)I_b(\tau),$$

$$\frac{dE_0^-(\tau)}{d\tau} = -\omega_0\gamma_2 E_0^+(\tau) + (2-\omega_0\gamma_1)E_0^-(\tau) - 2\pi(1-\omega_0)I_b(\tau), \quad (18)$$

where

$$E_0^+(\tau) = \frac{I_0^+(\tau)}{4}; \ E_0^-(\tau) = \frac{I_0^-(\tau)}{4}.$$

We will now examine more closely the case N = 1, which (unlike N = 0) yields the functions $I_0^+(\tau)$, $I_1^+(\tau)$, $I_0^-(\tau)$, $I_1^-(\tau)$ and thus also the thermal radiation flux q with sufficient accuracy, as will be shown here. When N = 1, the coefficients matrix (10) yields 16 coefficients. By virtue of properties 1 to 3,

$$\gamma_{+1+0} = \gamma_{+0+1} = -\gamma_{-1-0} = -\gamma_{-0-1} = \gamma_3,$$
$$\gamma_{-0+1} + \gamma_{+0+1} = 0, \quad \text{or} \quad \gamma_{-0+1} = -\gamma_3,$$
$$\gamma_{+0-1} + \gamma_{-0-1} = 0, \quad \text{or} \quad \gamma_{+0-1} = \gamma_3,$$
$$\gamma_{-1+0} = -\gamma_{+1-0} = \gamma_4, \ \gamma_{+1-1} = \gamma_{-1+1} = \gamma_5,$$
$$\gamma_{+1+1} = \gamma_{-1-1} = \gamma_6. \tag{18}$$

On the basis of the normalization (12), moreover, we have

$$\gamma_3 + \gamma_4 = 3\int_0^1 d\mu \left[\int_0^1 P_1(2\mu' - 1)P(\mu, \mu')\,d\mu' + \int_{-1}^0 P_1(2\mu' + 1)P(\mu, \mu')\,d\mu'\right] = 6\int_0^1 d\mu \int_{-1}^1 \mu' P(\mu, \mu')\,d\mu' \cdot$$

$$-3(\gamma_1 - \gamma_2) = 6\sum_{j=0}^{l} a_j \int_0^1 d\mu \int_{-1}^1 P_j(\mu)\mu' P_j(\mu')\,d\mu'$$

$$-3(\gamma_1 - \gamma_2) = 2\left[a_1 - 3(\gamma_1 - 1)\right], \tag{20}$$

$$\gamma_5 + \gamma_6 = 9\int_{-1}^0 d\mu \int_0^1 P_1(2\mu + 1)P_1(2\mu' - 1)P(\mu, \mu')\,d\mu' + 9\int_0^1 d\mu \int_0^1 P_1(2\mu - 1)P_1(2\mu' - 1)P(\mu, \mu')\,d\mu'$$
$$= 18\sum_{j=0}^{l} a_j \int_0^1 d\mu' \int_{-1}^1 (2\mu' - 1)P_j(\mu')\mu P_j(\mu)\,d\mu - 3(\gamma_3 - \gamma_4)$$
$$= 2a_1 - 3(\gamma_3 - \gamma_4). \tag{21}$$

When N = 1, therefore, it suffices to know three coefficients which depend on the dispersion indicatrix:

$$\gamma_1 = \int_0^1 d\mu \int_0^1 P(\mu, \mu')\,d\mu', \tag{22}$$

$$\gamma_3 = 3\int_0^1 d\mu \int_0^1 P_1(2\mu' - 1)P(\mu, \mu')\,d\mu', \tag{23}$$

$$\gamma_6 = 9\int_0^1 d\mu \int_0^1 P_1(2\mu - 1)P_1(2\mu' - 1)P(\mu, \mu')\,d\mu', \tag{24}$$

inasmuch as all other coefficients are expressed in terms of these.

Taking into account the relations derived here, an analytical expression for q is most conveniently obtained by reducing the system (6) to

$$\frac{de_1(\tau)}{d\tau} = A_1 e_3(\tau) + A_2 e_4(\tau) + 24\pi(1 - \omega_0)I_b(\tau),$$
$$\frac{de_2(\tau)}{d\tau} = A_3 e_3(\tau) + A_4 e_4(\tau) - 8\pi(1 - \omega_0)I_b(\tau),$$
$$\frac{de_3(\tau)}{d\tau} = A_5 e_1(\tau) + A_6 e_2(\tau),$$
$$\frac{de_4(\tau)}{d\tau} = A_7 e_1(\tau) + A_8 e_2(\tau), \tag{25}$$

where

$$e_1(\tau) = I_0^+(\tau) - I_0^-(\tau); \ e_2(\tau) = I_1^+(\tau) + I_1^-(\tau);$$
$$e_3(\tau) = I_0^+(\tau) + I_0^-(\tau); \ e_4(\tau) = I_1^+(\tau) - I_1^-(\tau);$$
$$A_1 = 3(\omega_0 - 1);$$
$$A_2 = \frac{1}{2}\left[3\omega_0(\gamma_3 - \gamma_4) - \omega_0(\gamma_6 - \gamma_5) + 6\right];$$
$$A_3 = 1 - \omega_0; \tag{26}$$

TABLE 2. Comparison between Radiation Flux q Calculated to the N = 1 Approximation with the Data in [6] (M) and with the N = 0 Approximation.

τ_0	q_M when $\tau=\tau_0$ from [6]	$q_{N=1}$ when $\tau=\tau_0$ according to [32]	$\frac{q_{N=1}-q_M}{q_M}$, %	$q_{N=0}$ when $\tau=\tau_0$	$\frac{q_{N=1}-q_{N=0}}{q_{N=0}}$, %
0,4	3,908	3,914	0,2	3,496	40,6
0,8	2,482	2,514	1,3	3,892	54,8
1,0	2,140	2,150	0,5	3,125	45,4
1,2	1,772	1,777	0,3	2,466	38,8
1,6	1,238	1,243	0,4	1,492	20,1
2,0	0,836	0,838	0,2	0,887	5,8

$$A_4 = \frac{1}{2}\left[\omega_0(\gamma_6-\gamma_5)-\omega_0(\gamma_3-\gamma_4)-6\right];$$

$$A_5 = \frac{1}{2}\left[3\omega_0(\gamma_1-\gamma_2)-2\omega_0\gamma_3-6\right];$$

$$A_6 = \frac{1}{2}\left[3\omega_0(\gamma_3+\gamma_4)-\omega_0(\gamma_6+\gamma_5)+6\right]; \tag{27}$$

$$A_7 = \frac{1}{2}\left[2\omega_0\gamma_3-\omega_0(\gamma_1-\gamma_2)+2\right];$$

$$A_8 = \frac{1}{2}\left[\omega_0(\gamma_6+\gamma_5)-\omega_0(\gamma_3+\gamma_4)-6\right].$$

If coefficients a_j in the Legendre expansion of the dispersion indicatrix are known, then the formulas in [4] for the integrals of Legendre polynomials yield the following expressions for coefficients γ_1, γ_3, and γ_6:

$$\gamma_1 = \int_0^1 d\mu \int_0^1 P(\mu,\mu')\,d\mu' = \sum_{j=0}^{l} a_j \left|\int_0^1 P_j(\mu)\,d\mu\right|^2 = 1 + \frac{1}{4}a_1 + \frac{1}{64}a_3 + \frac{1}{256}a_5, \tag{28}$$

$$\begin{aligned}\gamma_3 &= 3\int_0^1 d\mu \int_0^1 P_1(2\mu'-1)P(\mu,\mu')\,d\mu' \\ &= 6\int_0^1 d\mu \int_0^1 \mu' P(\mu,\mu')\,d\mu' - 3\gamma_1 \\ &= 6\sum_{j=0}^{l} a_j \int_0^1 P_j(\mu)\,d\mu \int_0^1 \mu' P_j(\mu')\,d\mu' - 3\gamma_1 \\ &= 3\left(1-\gamma_1+\frac{1}{3}a_1\right),\end{aligned} \tag{29}$$

$$\begin{aligned}\gamma_6 &= 9\int_0^1 d\mu \int_0^1 P_1(2\mu-1)P_1(2\mu'-1)P(\mu,\mu')\,d\mu' \\ &= 18\int_0^1 d\mu \int_0^1 \mu P_1(2\mu'-1)P(\mu,\mu')\,d\mu' - 3\gamma_3 \\ &= 9\left[\gamma_1 - 2\left(1+\frac{1}{3}a_1\right)\right] + 36\left(\frac{1}{4}+\frac{1}{9}a_1+\frac{1}{64}a_2+\frac{1}{2304}a_4\right).\end{aligned} \tag{30}$$

In these formulas we have retained only $l = 5$ terms of the indicatrix expansions, because such dispersion indicatrices are most often of interest.

In order to compare our method with other methods of calculating the radiation flux, system (25) was solved analytically for the case $\omega_0 = 0.5$ with a spherical and with the simplest elongated dispersion indicatrix $P(\theta) = 1 + \cos\theta$.

Let the boundary conditions for Eq. (1) be

$$I(0,\mu)|_{\mu>0} = E_1,\quad I(\tau_0,\mu)|_{\mu<0} = E_2. \tag{31}$$

TABLE 3. Comparison between Reflectivity Values and between Transmittivity Values for a Layer, Obtained in the Differential-Difference Approximation with an Isotropic Intensity Distribution in the Flux (g), by the Moments Method in the First Approximation (M), in an Approximation Analogous to the Eddington Approximation (E), and in the N = 1 Approximation

Reflectivity

τ_0	R_g	$R_{N=1}$	$\frac{R_g-R_{N=1}}{R_{N=1}}$, %	R_M	$\frac{R_M-R_{N=1}}{R_M}$, %	R_E	$\frac{R_E-R_{N=1}}{R_{N=1}}$, %
0,1	0,044	0,042	4,7	0,041	2,4	0,037	11,9
0,5	0,134	0,115	16,5	0,111	3,6	0,123	6,9
1,0	0,163	0,130	25,3	0,136	4,4	0,158	21,5
2,0	0,169	0,142	19,0	0,146	2,6	0,171	20,4

Transmittivity

τ_0	D_g	$D_{N=1}$	$\frac{D_g-D_{N=1}}{D_{N=1}}$, %	D_M	$\frac{D_M-D_{N=1}}{D_M}$, %	D_E	$\frac{D_E-D_{N=1}}{D_{N=1}}$, %
0,1	0,860	0,866	0,7	0,867	0,1	0,879	1,5
0,5	0,491	0,531	9,4	0,530	0,2	0,530	0,2
1,0	0,236	0,307	23,1	0,308	0,3	0,284	7,5
2,0	0,058	0,111	47,7	0,109	1,8	0,083	25,2

Through transformations analogous to those performed on (3), we arrive at

$$I_0^+(0) = 4\pi E_1,\ I_1^+(0) = 0,$$
$$I_0^-(0) = 4\pi E_2,\ I_1^-(0) = 0.$$

The boundary conditions for system (25) with $E_1 = 2$ and $E_2 = 0$ are then

$$[e_2(\tau) + e_4(\tau)]|_{\tau=0} = 0,\quad [e_1(\tau) + e_3(\tau)]|_{\tau=0} = 16\pi,$$
$$[e_2(\tau) - e_4(\tau)]|_{\tau=\tau_0} = 0,\quad [e_3(\tau) - e_1(\tau)]|_{\tau=\tau_0} = 0.$$

They signify that there is no radiation when $\tau = \tau_0 (e_2(\tau) - e_4(\tau) = 2I_1^-(\tau),\ e_3(\tau) - e_1(\tau) = 2I_0^-(\tau))$ and that the radiation is diffusive when $\tau = 0 (e_2(\tau) + e_4(\tau) = 2I_1^+(\tau))$, while its intensity is determined by the magnitude of $I_0^+(0) (e_1(\tau) + e_3(\tau) = 2I_0^+(\tau))$.

The expression for the thermal radiation flux is, according to (17),

$$q = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} + c_3 e^{\lambda_3 \tau} + c_4 e^{\lambda_4 \tau}, \tag{32}$$

where coefficients c_i have been obtained from the boundary conditions and its values are listed for various values of τ_0 in Table 1. The characteristic values are the roots of the characteristic equation of system (25):

$$\lambda_{1,2} = \pm \sqrt{\frac{a + \sqrt{a^2 - 4b}}{2}},\quad \lambda_{3,4} = \pm \sqrt{\frac{a - \sqrt{a^2 - 4b}}{2}},$$

where

$$c = A_1 A_5 + A_2 A_7 + A_3 A_6 + A_4 A_8,$$
$$b = (A_5 A_8 - A_6 A_7)(A_1 A_4 - A_2 A_3). \tag{32a}$$

Calculations of the thermal radiation flux for $\tau = \tau_0$ and the simplest forward elongated dispersion indicatrix are compared in Table 2 with the calculations in [6] and with the results of the zeroth approximation. Evidently, the results according to formula (32) agree closely with those in [6], where the problem has been solved by numerical methods basically to the same approximation, while a comparison with the exact solution indicates that the method proposed in [6] is very accurate. The zeroth approximation, according to Table 2, does not yield the necessary accuracy in the calculation of radiation fluxes.

With the aid of Eq. (32), one can determine the radiation characteristics of a layer. On the basis of (31), a hemispherical radiation flux incident on a layer with $\tau = 0$ is equal to πE_1. A radiation flux q(0) represents the difference between the flux incident on the layer and the flux reflected from the layer, if there is no radiation when $\tau = \tau_0$. Therefore, the transmittivity and the reflectivity of a layer can be calculated according to these respective formulas:

$$D = \frac{q(\tau_0)}{\pi E_1},$$
$$R = \frac{\pi E_1 - q(0)|_{E_2=0}}{\pi E_1} = 1 - \frac{q(0)|_{E_2=0}}{\pi E_1}. \quad (33)$$

Calculations according to formula (33) and the corresponding formulas in [5], where a solution has been obtained on the basis of the differential-difference approximation for an isotropic intensity distribution in the flux, are compared in Table 3 with the solution by the moments method in the first approximation [6] and with the solution in the approximation analogous to the Eddington approximation and also shown in [6]. The latter solution had been obtained on the basis of the one-dimensional problem of radiation through a layer [7, 8], by changing $\tau_0 \to \sqrt{3\tau_0}$.

According to Table 3, the results of the N = 1 approximation agree closely with the solution by the moments method in the first approximation, which in turn ensures a satisfactory accuracy in the calculation of radiation fluxes.

NOTATION

A_i	are the coefficients in system (25) defined for formulas (27);
a, b	are the expressions defined in (32a);
a_i	are the coefficients in the Legendre polynomial expansion of the dispersion indicatrix;
D	is the transmittivity of a layer;
E_0^+, E_0^-	are the radiation fluxes in opposite directions;
$I(\tau, \mu)$	is the radiation intensity;
$I_b(\tau)$	is the radiation intensity of a perfectly black body;
$e_i(\tau)$	are the quantities defined by formulas (26) (i = 1, 2, 3, 4);
R is the	is the reflectivity of a layer;
q	is the radiation flux;
α_{ik}, β_{ik}, γ_{ik}	are the coefficients defined by formulas (7), (8), and (10);
λ_i	are the roots of the characteristic equation of system (25);
ε_i	is the emissivity of a boundary surface (i = 1, 2);
ρ_i	is the reflectivity of a boundary surface (i = 1, 2);
τ	is the optical thickness (density);
θ, φ	are the angles;
$\eta = \cos\theta$;	
ω_0	is the albedo of a single dispersion.

LITERATURE CITED

1. B. Davison and J. B. Sykes, Neutron Transport Theory, Oxford University Press (1957).
2. S. Chandrasekar, Transmission of Radiant Energy [Russian translation], Izd. Inostr. Lit., Moscow (1953).
3. V. N. Adrianov, in: Heat Transfer, Hydrodynamics, and Thermophysical Properties of Matter [in Russian], Izd. Nauka, Moscow (1968), p. 123.
4. Whittacker and Watson, Study Course in Modern Analysis [Russian translation], Gosizdat, Moscow (1963), Part 2.
5. V. N. Adrianov, in: Heat Transfer, Hydrodynamics, and Thermophysical Properties of Matter [in Russian], Izd. Nauka, Moscow (1968), p. 139.
6. Yu. A. Popov, Teplofiz, Vys. Temp., No. 1 (1969).
7. V. V. Sobolev, Energy Radiation through the Atmosphere of Stars and Planets [in Russian], Gostekhizdat (1956).
8. M. M. Gurevich, Phys. Zeitschr. [German], 31, 753 (1930).